1. Introduction Modeling motion of mechanical system can be made in several equivalent ways. For instance, Lagrange multipliers method, Null space method and Magi’s method are some of them [1,2,3]. In each of this methods a set of Differential-Algebraic-Equations (DAE) results. If these set of DAE of motion does not use explicitly the position and velocity equations associated to the constraints, then it leads to the problem of stability at the position and velocity level of the constraints. The strategies generally used to overcome this problem are the Coordinate Partitioning Method [2], the Baumgarte’s Stabilization Method [4], the Augmented Lagrangian formulation or Mass-orthogonal projections of position and velocity vectors [5]. Moreover, in addition to the stability problems that may happen while simulating constrained mechanical systems, problems related to the presence of redundant constraints is also unavoidable in practice. In the presence of more equations than strictly needed the Jacobian matrix becomes rank deficient. This can be observed when some of the equations are dependent in the remaining ones. This makes the leading matrix, for example in Lagrangian Index one equation, singular. The leading matrix can also be singular when the mass matrix is singular. A Singular Mass matrices may appear when more than six coordinates are used to define the position of a rigid body in [2].The Jacobian matrix can also be rank deficient when the mechanical system reaches a configuration in which there is a sudden change in the number of degree of freedom. For instance, a slider crank mechanism [5] reaches a singular configuration when both the two links are at vertical position. In this position both links overlap and the mechanism has not one but two degree of freedom that corresponds to two possible motions that the mechanism can undergo. This paper presents a brief discussion on modeling a mechanical system with Index one Lagrange’s equation of motion in the presences of redundant holonomic constraints. As the Index one Lagrange’s equation of motion does not include the position and velocity constraint equations explicitly, it does not provide any solution for the constraint violation problem. Therefore, a technique useable to minimize the constraint violation errors is still required. To this end Baumgarte’s method of stabilization is included in the discussion. The main purpose of this paper is to show the application of Generalized Inverse of a matrix in finding the acceleration, the Lagrangian multipliers and as a result, the constraint forces of a mechanical system with a holonomic constraint imposed. This is discussed based on properties of Generalized Inverse of a Matrix and Singular Value Decomposition. The advantage of this method is that, it can handle singular mass matrices and redundant constraints. That is, the mass matrix can in general be assumed to be square symmetric and positive semidefinite. The positive definiteness of the mass matrix is not required in the application of this method. Redundant constraints are handled in the solution of system of equations of motion and the problems that involve singular configurations and redundant constraints, and other problems associated with changing the number of degrees of freedom can be managed using the method developed in this paper. 2. Constructing Equation of

Motion for a Holonomic Constrained Mechanical Systems In this section we will discuss a general approach of constructing equation of motion of a mechanical system in which a holonomic constraint is involved based on the literature. Let be generalized coordinate of a system and suppose that the system is subjected to holonomic constraints given by: . (1) Let be the Lagrangian of the system where and are respectively the kinetic and potential energy of the system. Then the Lagrange's equation of motion of the system can be given by [3, 6, 7]: . (2) Where generalized external is force, and is Lagrange's multiplier. The kinetic energy of a multibody system can be written in the form [1,2,7]: , (3) where is the mass matrix of the system which is assumed to be symmetric and positive definite square matrix. Since we have: (4) Then equation (2) can be written as: (5) where . Putting equation (5) becomes: . (6) The position, velocity and acceleration vectors in Equation (5) must satisfy the corresponding constraint equations: , (7a) (7b) (7c) Equation (7a) − (7c) and (6) together constitutes DAE of Index 3 [7] with and as unknowns. However, if only equations (7c) and (6) are considered, the following index 1 DAE system equivalent to an ODE system is obtained: This system is said to be [2] Lagrange's Index one system of dynamic equations. (8), where The system of differential equations (8) presents a constraint stabilization problem. As only the acceleration constraint equations have been imposed, the positions and velocities provided by the integrator suffer from a drift phenomenon. Some solutions to this problem are the Baumgarte's stabilization method [4] and the mass orthogonal projections of position and velocity vectors [2,5]. We will discuss Baumgarte's method of constraint stabilization. To secure Baumgartes stabilization of the constraint equation we replace [4] in equation 7c by: , (9) Where and are appropriately chosen constants. After replacing we obtain: Substituting , we obtain: (10) Considering equations (6) and (10) we obtain the equation of the dynamic system to be: (11) Equation (11) constitutes index one Ordinary Differential Equations, with the required values and .But still after Baumgarte's stabilization method is applied to the system, there could be a problem of redundant constraints and singular mass matrices. Let us see the following example that describes how singular mass matrix may happen. A Singular Mass matrices may appear when more than six coordinates are used to define the position of a rigid body in . When Euler parameters or natural coordinates are used this is always the case. With natural coordinates [2,9] the constant inertia matrix of a rigid body requires that the body be defined with two points and two unit vectors (or a similar configuration, for instance with four non-coplanar points). If this body has additional points and unit vectors, the corresponding rows and columns of the inertia matrix have null values, making this matrix positive semidefinite. In the case of redundant constraints in equation (1) the Jacobian matrix does not have full rank. In this situation we can obtain the resultant reaction force of the constraints but not [2,9,10] each of the , =1,2,… .Let us consider the following two cases: 1. Assuming that is positive definite and the Jacobean Matrix has full rank, that is: . Then the value of from equation (6) can be obtained to be: , (12) and substituting(12) into (10) we obtain , (13) which yields: . (14) The value of can be obtained from (14) and then putting its value in (12) the corresponding value of can be obtained. That is: (15) 2. Assume that M is positive definite and the Jacobian Matrix is rank Deficient. As it is discussed above, redundant constraints in equation (1) can be reflected by the fact that some of the equations are dependent in the remaining ones. This lack of equation independence in the system (1) may lead to a rank deficiency in the Jacobian matrix and an over constrained system of linear equations (more equations than unknowns) which will not have a solution that satisfy all the equations. In such situations, suppose we need to find each of the Lagrangian Multipliers , =1,2,… and then each of the constraint forces of a mechanical system. That is, assuming that , and for any convenient vector, let The general solution, by the Minimum Norm Solution method using Generalized Inverse of a matrix, of[8]: , (16) is given by: , (17) where is by identity matrix and is arbitrary Lagrangian multiplier vector, represents the orthogonal projection matrix in the null space of .That is Equation (17) can be decomposed [7,8] as: (18) where is the minimum norm solution that minimizes and is a vector in the kernel of . With regard to (17) we have the following two cases: a) If has a full rank then, and hence equation (17) reduces to , since I − = 0. Therefore in this case we have a unique solution which is called the pseudoinverse solution [8].Note that, if B is a square matrix and has a full rank then = and in this case the nullspace of contains only the zero vector. b) If is rank deficient we apply the method of in which case can be calculated as where and are by and by square orthogonal matrices respectively and has the same size as and is by matrix. The non-square matrix [1]has non-zero elements only on its diagonal and therefore, the calculation of its generalized inverse is trivial [8]. Note that, is based on a theorem from linear algebra which says that a rectangular matrix can be broken down into the product of three matrices - an orthogonal matrix , a diagonal matrix , and the transpose of an orthogonal matrix . The theorem is usually presented something like this: ,where the columns of are orthonormal eigenvectors of , the columns of are orthonormal eigenvectors of , and is a diagonal matrix containing the square roots of eigenvalues from or in descending order. The generalized inverse of B is [7] where the relation valid for orthogonal matrices have been used and: The sub-matrix contains the reciprocal of the non-zero singular values along the principal diagonal. In this case, is rank deficient, then equation (16) has an infinite number of solutions given by equation (17). In summary: i).When we apply the method discussed above to Equation (13) assuming that is rank deficient we obtain to be: where is an arbitrary Lagrangian multiplier. By pressing on we can obtain infinite solutions for .Substituting we obtain the minimum norm solution: that minimizes ii).We can obtain from equation (15) as follows. When we closely look at equation (15) we observe that should exist for the values of be obtained. The existence may fail in case the Jacobian matrix is rank deficient. In this instance one method to obtain is to make use of the assumption that is positive definite that grantees the diagonalizability of . Let the mass matrix resulting from unconstrained mechanical system be positive definite and the acceleration of the unconstrained system be denoted by . Then we have and referring to equation (15) we have: (19) From the fact that is positive definite we obtain and hence . Then equation (19) becomes: (20) Putting equation (20) becomes (21) From the properties of generalized inverse of a matrix we have . Hence equation (21) becomes: (22) It can be seen in (22) that the values of is independent of whether the coefficient matrix in equation (8) or (11) is rank deficient or not. Moreover, can be calculated as explained above using method. The next point will be on how to obtain : Since is assumed to be symmetric and positive definite [8] it has a unique square root , such that . In order to find matrix first we need to diagonalize matrix . That is, we need to find a matrix which consists of the orthonormal eigenvectors of and a diagonal matrix with its diagonal elements the corresponding eigenvectors of such that . Next, since all the eigenvalues of are positive we can write where is obtained from by replacing all the diagonal elements with its square root. Finally we calculate to obtain the square root of . . Indeed, .The inverse of the square root of , denoted by is obtained to be: , Where for orthonormal matrix , is used. iii).For the case of positive semidefinite mass matrix, from Equation (6) we obtain: , where is an arbitrary acceleration vector of the system. As in the above choosing gives the minimum norm solution for the acceleration of the system to be: It is to be noted that the resultant constraint force can always be obtained irrespective of the rank of matrix . 3. Acceleration and Lagrangian Multipliers of Mechanical System in the case of Redundant Constraints and Positive Semidefinite Mass Matrices We can also apply the properties of generalized inverse of a matrix to obtain the acceleration and Lagrange's multipliers, at the same time, from equation of a constrained mechanical system given by (8) or (11). Let as assume the mass matrix , resulting from the unconstrained mechanical system is symmetric positive semidefinite square matrix. We dropped the assumption that is positive definite. is in general considered to be positive semidefinite. Consider equation (11) and let: (23) Then equation (11) can be written in the form: (24) With the same method used above the General Solution of equation (24) becomes: (25) where is an Identity matrix of and is an arbitrary vector of size .The Minimum Norm solution that minimizes . The main advantage of this formulation lies in the fact that ,the generalized inverse of , always exists provided that .In other words, the formulation is applicable even if the mass matrix is singular and the Jacobian matrix is rank deficient. Let us write this result as follows. Result 1: The General Solution of equation of motion of a constrained mechanical system described by equation (11), whether the matrix M that arises in the unconstrained system is singular, whether the constraint is redundant is given by: (26) It is clear that, because of the arbitrary vector in equation (26) the solution of and λ is not necessarily unique. However, if in equation (23) has a full rank then and hence the solution becomes unique. We can state this result as follows: Result 2: When the matrixhas a full rank then the General Solution of the mechanical system given by equation (11) becomes unique and is given by: (27) The next logical question could be what are the necessary and sufficient conditions for to have a full rank so that we can have a unique solution? Let us investigate it as follows: a).Assume that is symmetric positive semidefinite and has full rank. If is positive definite on the kernel of (i.e. and implies that ), then is nonsingular. Proof: We show that for . Indeed let ,, . Then implies: . (28) It then follows that yielding since . From we obtain since for and . Substituting x = 0 and nothing that has a full rank one can obtain from equation (28) that .Hence we showed that . This shows that is nonsingular. (Sufficient condition for to be nonsingular) b).On the other hand suppose is nonsingular, and . Proof. We want to show that . On the contrary let then since is positive semidefinite we obtain yielding letting and nothing that we can see that for a non zero vector . This contradicts the supposition that is nonsingular. (Necessary condition) The result obtained from a) and b) can be written as follows: Result 3: Let be symmetric positive semidefinite and has full rank. Matrix has a full rank if and only if is positive definite on the kernel of . Remark: 1. In Result 3 above the condition that is positive definite on the kernel of can be relaxed to “the mass matrix is definite on the kernel of ”. This is because in the proof we used only. If M is indefinite the following example shows that is singular even though the Jacobian matrix has a full rank. From the above matrix it can be seen that the Jacobiam matrix has a full rank and is indefinite but is singular. 2. In equation (26) the Minimum Norm Solution is always unique and is given by , obtained by putting . Example: This problem is adapted from the exercises given in chapter 2 of [11].A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius as shown in figure 1. The only external force is that of gravity. If the smaller cylinder starts rolling from rest on top of the bigger cylinder, find the acceleration and each of the constraint forces before the hoop falls off the cylinder. Fig. 1 Solution: Two equations of constraints: , . The generalized coordinates are and . The first equation is from the fact that as long as the hoop is touching the cylinder the center of mass of the hoop is exactly away from the center of the cylinder. The second one comes from no slipping: . Where is the angle makes with the vertical and is the angle makes with the vertical. The kinetic energy is the sum of the kinetic energy of the center of mass of the hoop and the kinetic energy of the hoop about the cylinder given by: . The potential energy is the height above the center of the cylinder and is given by: . The Lagrangian is given by , and from the Lagrange’s equations: and we obtain: (29) The constraint equations at the acceleration level are given by: (30) Now from equation (13) and using (29) and (30) we obtain: (31) Note that is a singular matrix. Now using the method developed on equation (17) we have the General Solution of the Lagrangian multipliers given by: (32) Where is arbitrary non-zero vector. It can easily be shown that . Hence the General Solution becomes: (33) Note that: The Minimum Norm Solution is unique and is given by . On the other hand combining the equations of unconstrained mechanical system, Equation (29), the constraint equation (30) and using equation (11) ,(In fact in this example no constraint stabilization method is used) together we obtain the equation of the constrained system as: From which the General

Solution of the system becomes Numerically, let us suppose that R=1m, r=0.2m, m=2kg, g=9.8m/s2, is arbitrary non-zero vector then From which it can be seen that: , where is an arbitrary constant. One can verify that the values we obtained for the Lagrangian multipliers here and in (33) are the same for m=2kg 1. Each of the constraint forces is given by: 2. The acceleration of the system is given by: From which it can be solved that: where , and are arbitrary constants that can be determined based on initial conditions. If the hoop starts from rest then From these initial conditions we obtain: 4. Conclusions The application of the methods we used here can equivalently be applied to other methods of modeling mechanical systems mentioned in the introductory part of this paper including nonholonomic constraint systems. It must also be noted that, even though we may come up with infinite number of equations of motion and infinite number of Lagrangian multipliers of a mechanical system, in all practical purposes, we make use of the minimum norm solutions which is always unique. The calculation of generalized inverse of a matrix, especially with one or more variable entries seems to be expensive, but if all the entries are scalars, obtaining the generalized inverse is not that expensive. In the second case one can also use MATLAB and other software. [1] S is rectangular diagonal matrix, which is an m-by-n matrix with only the entries of the form di,i possibly non-zero.